DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN The Netherlands

A NEW CHARACTERIZATION OF THE UNIT BALL OF H^2

R.A. Kortram

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN
Toernooiveld
6525 ED Nijmegen
The Netherlands

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Abstract

We derive a new expression for the norm of H^2 functions; we present some well-known results in a different setting.

Introduction

In 1915, Pick [3] proved the following result

Theorem 1 *Let g be an analytic function on the unit disc Δ in the complex plane. Then $|g(z)| \leq 1$ for all $z \in \Delta$ if and only if for all $n \in \mathbb{N}$, for all sequences $z_1, z_2, \ldots, z_n$ in Δ and for all sequences $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have*

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)\overline{g(z_l)}}{1 - z_k \overline{z}_l} \lambda_k \overline{\lambda_l} \geq 0. \tag{1}$$

Ahlfors [1] page 3, gives an elegant proof of this characterization of the unit ball of H^∞.

In this note we shall present a characterization of the unit ball of H^2. Our main tool will be an explicit solution of the "Minimal Interpolation Problem" for H^2. [2] page 141. As a byproduct we obtain a new proof of Pick's theorem.

Description of the main result

Let $z_1, z_2, \ldots, z_n$ be a sequence in Δ, and let b be the Blaschke product generated by this sequence:

$$b(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z}_j z}. \tag{2}$$

We shall prove that the following conditions are equivalent for continuous functions f on Δ.

1) f lies in the unit ball of H^2.

2) for every $n \in \mathbb{N}$ and for every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points in Δ we have

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k) \cdot \overline{f(z_l)}}{1 - z_k \overline{z}_l} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}} \leq 1. \tag{3}$$

Preliminaries

For mutually distinct points $z_1, z_2, \ldots, z_n$ in Δ and for $w_1, w_2, \ldots, w_n$ in $\mathbb{C}$ we define

$$\Lambda = \{f \in H^2 : f(z_j) = w_j,\ j = 1, 2, \ldots, n\}.$$

Λ is not empty; it contains the Lagrange interpolation polynomial

$$\lambda(z) = l(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k) \cdot l'(z_k)},$$

where $l(z) = \prod_{j=1}^{n} (z - z_j)$.

In the context of H^p spaces it is more natural to work with the Blaschke interpolation function

$$\beta(z) = b(z) \sum_{k=1}^{n} \frac{1 - \overline{z}_k z}{z - z_k} \cdot \frac{w_k}{b'(z_k)(1 - |z_k|^2)},$$

with $b(z)$ defined as in (2). Of course $\beta \in \Lambda$. However, for our purposes we are better off with

$$\varphi(z) = b(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k) b'(z_k)}. \tag{4}$$

$\varphi \in \Lambda$, and φ is analytic on some neighbourhood of $\overline{\Delta}$. Λ is a hyperplane in H^2. With φ and b defined as in (4) and (2) we have

$$\Lambda = \{\varphi + b \cdot g;\ g \in H^2\}.$$

Theorem 2 *φ is the unique solution of the "Minimal Interpolation Problem", i.e. for every $f \in \Lambda \backslash \{\varphi\}$ we have $\|f\|_2 > \|\varphi\|_2$.*

Proof: It suffices to show that $\varphi \perp (f - \varphi)$ for every $f \in \Lambda$ (since under those circumstances $\|f\|^2 = \|\varphi\|^2 + \|f - \varphi\|^2$).
From the decomposition $f = \varphi + b.g$ we have

$$\begin{aligned} \langle f - \varphi, \varphi \rangle &= \langle b \cdot g, \varphi \rangle = \frac{1}{2\pi} \int_0^{2\pi} b(e^{it}) g(e^{it}) \overline{\varphi(e^{it})} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} b(e^{it}) g(e^{it}) \overline{b(e^{it})} \sum_{k=1}^{n} \frac{\overline{w}_k}{(e^{-it} - \overline{z}_k) \cdot \overline{b'(z_k)}} dt. \end{aligned}$$

Note that $|b(e^{it})|^2 = 1$. Thus

$$\begin{aligned} \langle f - \varphi, \varphi \rangle &= \sum_{k=1}^{n} \frac{\overline{w}_k}{2\pi \overline{b'(z_k)}} \int_0^{2\pi} g(e^{it}) \frac{e^{it}}{1 - e^{it} \overline{z}_k} dt \\ &= \sum_{k=1}^{n} \frac{\overline{w}_k}{\overline{b'(z_k)}} \cdot \frac{1}{2\pi i} \int_{\Gamma} \frac{g(z)}{1 - \overline{z}_k z} dz = 0, \end{aligned}$$

because the integrand is analytic on Δ.

It will be convenient to have an explicit expression for $\|\varphi\|_2$.

$$\begin{aligned}\|\varphi\|_2^2 &= \frac{1}{2\pi}\int_0^{2\pi}|\varphi(e^{it})|^2dt = \frac{1}{2\pi}\sum_{k=1}^n\sum_{l=1}^n\frac{w_k\overline{w}_l}{b'(z_k)\overline{b'(z_l)}}\int_0^{2\pi}\frac{dt}{(e^{it}-z_k)(e^{-it}-\overline{z}_l)}\\ &= \frac{1}{2\pi i}\sum_{k=1}^n\sum_{l=1}^n\frac{w_k\overline{w}_l}{b'(z_k)\overline{b'(z_l)}}\int_\Gamma\frac{dz}{(z-z_k)(1-\overline{z}_l z)} = \sum_{k=1}^n\sum_{l=1}^n\frac{w_k\overline{w}_l}{1-z_k\overline{z}_l}\;\frac{1}{b'(z_k)\overline{b'(z_l)}}.\end{aligned}$$

There are of course many other expressions for $\|\varphi\|_2$.

Theorem 3

$$\|\varphi\|_2 = \max\left\{\left|\sum_{k=1}^n\frac{w_k f(z_k)}{b'(z_k)}\right| : f\in H^2, \|f\|_2\le 1\right\}.$$

Proof:

$$\sum_{k=1}^n\frac{w_k f(z_k)}{b'(z_k)} = \frac{1}{2\pi i}\int_\Gamma\frac{f(z)\varphi(z)}{b(z)}dz,$$

hence by Schwarz's inequality we have

$$\left|\sum_{k=1}^n\frac{w_k f(z_k)}{b'(z_k)}\right| \le \frac{1}{2\pi}\int_0^{2\pi}|f(e^{it})|\cdot|\varphi(e^{it})|dt \le \|f\|_2\cdot\|\varphi\|_2 \le \|\varphi\|_2.$$

Equality holds for the function $f : z \to \frac{1}{\|\varphi\|_2}\sum\limits_{k=1}^n\frac{\overline{w}_k}{(1-\overline{z}_k z)\overline{b'(z_k)}}$.

An immediate result from Theorem 2 is

Corollary For every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points of Δ we have

$$\sum_{k=1}^n\sum_{l=1}^n\frac{1}{1-z_k\overline{z}_l}\cdot\frac{1}{b'(z_k)\overline{b'(z_l)}}\le 1.$$

Proof: Take $w_1 = w_2 = \ldots = w_n = 1$. Then $1\in\Lambda$ and since

$$\|1\|_2 = 1,$$

we have

$$1 \ge \|\varphi\|_2^2 = \sum_{k=1}^n\sum_{l=1}^n\frac{1}{1-z_k\overline{z}_l}\cdot\frac{1}{b'(z_k)\overline{b'(z_l)}}.$$

The equality sign certainly occurs if $0\in\{z_1, z_2, \ldots, z_n\}$:

$$1 = \varphi(0)^2 \le \frac{1}{2\pi}\int_0^{2\pi}|\varphi(e^{it})|^2dt = \|\varphi\|_2^2 = \sum_{k=1}^n\sum_{l=1}^n\frac{1}{1-z_k\overline{z}_l}\cdot\frac{1}{b'(z_k)\overline{b'(z_l)}}.$$

If $0 \notin \{z_1, z_2, \ldots, z_n\}$ there is strict inequality:
Because of the uniqueness of φ there can be equality only if

$$b(z) \sum_{k=1}^{n} \frac{1}{(z - z_k) b'(z_k)} = 1.$$

In this identity for rational functions we let $z \to \infty$. Since $z_j \neq 0$, $\lim\limits_{z\to\infty} b(z)$ has a finite value. Therefore, the left hand side has limit zero.

The fact that $\varphi \in \Lambda$ has an interesting reformulation. We start with a lemma.

Lemma 1 *The partial fraction decomposition of φ is*

$$\varphi(z) = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - \overline{z}_l z)(1 - \overline{z}_l z_k) b'(z_k) \overline{b'(z_l)}}. \tag{5}$$

Proof: An elegant way to prove this is to compute both sides of the following identity. For $z \in \Delta$ we have

$$\frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\zeta)}{1 - \overline{\zeta} z} \cdot \frac{d\zeta}{\zeta} = \overline{\frac{1}{2\pi i} \int_\Gamma \frac{\overline{\varphi(\zeta)}}{1 - \zeta\overline{z}} \cdot \frac{d\zeta}{\zeta}}.$$

The left hand side is equal to

$$\frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\zeta)}{\zeta - z} dz = \varphi(z),$$

while the right hand side is equal to the complex conjugate of

$$\frac{1}{2\pi i} \int_\Gamma \overline{b(\zeta)} \sum_{k=1}^{n} \frac{\overline{w}_k}{\overline{(\zeta - z_k) b'(z_k)}} \cdot \frac{1}{1 - \zeta\overline{z}} \cdot \frac{d\zeta}{\zeta},$$

i.e. to the complex conjugate of

$$\frac{1}{2\pi i} \int_\Gamma \frac{1}{b(\zeta)} \sum_{k=1}^{n} \frac{\overline{w}_k}{(1 - \overline{z}_k \zeta) \overline{b'(z_k)}} \frac{d\zeta}{1 - \overline{z}\zeta}.$$

Calculation of the residues at the points $z_1, z_2, \ldots, z_n$ lead to (5).

The condition $\varphi \in \Lambda$ implies that $\varphi(z_j) = w_j$, $j = 1, \ldots, n$ i.e.

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - \overline{z}_l z_j)(1 - \overline{z}_l z_k) b'(z_k) \overline{b'(z_l)}} = w_j.$$

This is equivalent to the assertion that the matrices

$$B = (\beta_{lk})$$

and its conjugate $\overline{B} = (\overline{\beta}_{lk})$ where

$$\beta_{lk} = \frac{1}{(1 - \overline{z}_l z_k) b'(z_k)}$$

are each others inverse, i.e. B and $\overline{B}$ are unitary.

Proof of the main result

Lemma 2 *Assume that f lies in the unit ball of H^2, and let a sequence of mutually distinct points $z_1, z_2, \ldots, z_n$ in Δ be given. Then (3) holds.*

Proof: Define $w_j = f(z_j)$. f lies in the hyperplane Λ and the element φ of Λ with minimal norm satisfies

$$\|\varphi\|_2 \leq \|f\|_2 \leq 1.$$

Use of the explicit expression for $\|\varphi\|_2$ leads to (3).

Lemma 3 *Assume that f is continuous and that f satisfies (3). We shall show that $f \in H^2$ and that $\|f\|_2 \leq 1$.*

Proof: We apply (3) for the case $n = 1$; an easy computation shows that

$$|f(z)| \leq \frac{1}{\sqrt{1-|z|^2}} \tag{6}$$

for every choice of $z \in \Delta$.

Let $0 < r < \rho < 1$, and let $z_1, z_2, z_3, \ldots$ be an enumeration of the rational points of $\overline{\Delta}_\rho$. For every n there is a function φ_n with

$$\varphi_n(z_j) = f(z_j), \quad j = 1, 2, \ldots, n$$

and

$$\|\varphi_n\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)\overline{f(z_l)}}{1 - z_k \overline{z}_l} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}} \leq 1.$$

Thus, φ_n lies in the unit ball of H^2, and so by lemma 2, we have for every sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$ in Δ

$$\sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\varphi_n(\zeta_k)\overline{\varphi_n(\zeta_l)}}{1 - \zeta_k \overline{\zeta}_l} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}} \leq 1.$$

It follows from (6) that

$$|\varphi_n(\zeta)| \leq \frac{1}{\sqrt{1-|\zeta|^2}},$$

hence the sequence $\varphi_1, \varphi_2, \ldots$ is uniformly bounded on $\overline{\Delta}_\rho$. Therefore, it contains a locally uniformly convergent subsequence φ_{n_j}. At the points $z_1, z_2, \ldots$ the subsequence converges to f. By the continuity of f and the fact that $\{z_1, z_2, \ldots\}$ is dense in Δ_ρ we see that

$$\lim_{n_j \to \infty} \varphi_{n_j} = f.$$

This shows that f is analytic on Δ_ρ for all $\rho < 1$. Because of uniform convergence on Γ_r we have

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt = \lim_{n_j \to \infty} \frac{1}{2\pi} \int_0^{2\pi} |\varphi_{n_j}(re^{it})|^2 dt \leq 1.$$

Thus, $f \in H^2$ and $\|f\|_2 \leq 1$.
Lemma 2 and lemma 3 together constitute a proof of the main result.

Corollary For $f \in H^2$ we define

$$\nu(f) = \sup \left\{ \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)\overline{f(z_l)}}{1 - z_k \overline{z}_l} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}} ; z_1, z_2, \ldots, z_n \text{ mutually distinct points of } \Delta \right\}.$$

Then $\nu(f) = \|f\|_2^2$.
Proof: Assume that $\nu(f) = 1$. Then by lemma 3: $\|f\|_2^2 \leq 1$. If $\|f\|_2^2 < \lambda^2 < 1$ for some λ, then we have $\left\|\frac{1}{\lambda} f\right\| < 1$ but $\nu\left(\frac{1}{\lambda} f\right) > 1$ which is impossible by lemma 2.
In a similar way we can show that $\|f\|_2^2 = 1$ implies that $\nu(f) = 1$. By the homogeneity of ν and $\| \quad \|_2^2$ it follows that for all $f \in H^2$: $\nu(f) = \|f\|_2^2$.

Pick's theorem

As an application of our results we shall give a proof of Pick's theorem.

Let g belongs to the unit ball of H^∞, and let $z_1, z_2, \ldots, z_n$ be a sequence of mutually distinct points in Δ. Let $w_1, w_2, \ldots, w_n$ be an arbitrary sequence of complex numbers. We consider the hyperplanes Λ and Λ_g where

$$\Lambda_g = \{f \in H^2 : f(z_j) = w_j \cdot g(z_j), j = 1, 2, \ldots, n\}.$$

Of course, if $f \in \Delta$, then $g \cdot f \in \Delta_g$, and by Theorem 2 applied to Λ_g we have

$$\|gf\|_2^2 \geq \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k g(z_k) \cdot \overline{w_l g(z_l)}}{1 - z_k \overline{z}_l} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}}.$$

Let φ be, as before, the element of Λ with smallest norm. From $\|g\|_\infty \leq 1$ we obtain

$$\|g\varphi\|_2 \leq \|\varphi\|_2.$$

Combination of these steps leads to

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k \overline{w}_l}{1 - z_k \overline{z}_l} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}} = \|\varphi\|_2^2 \geq \|g\varphi\|_2^2 \geq \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k \overline{w}_l g(z_k)\overline{g(z_l)}}{1 - z_k \overline{z}_l} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}}$$

i.e. to

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)\overline{g(z_l)}}{1 - z_k \overline{z}_l} \cdot \frac{w_k \overline{w}_l}{b'(z_k)\overline{b'(z_l)}} \geq 0$$

and since the sequence $w_1, w_2, \ldots, w_n$ is arbitrary we have for all choices of $\lambda_1, \lambda_2, \ldots, \lambda_n$

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)\overline{g(z_l)}}{1 - z_k \overline{z}_l} \cdot \lambda_k \overline{\lambda}_l \geq 0.$$

By the choice $n = 1$, $\lambda_1 = 1$ we see that the converse is trivial.

References

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[3] G. Pick; Über die Beschränkungen analytischer Funktionen, welche durch vorgeschriebene Werte bewirkt werden, *Math. Ann.* 77 (1915), 7-23.